MS221

Assignment Booklet I 2006B

Contents

Cut-off date

3	TMA MS221 01 (covering Block A)	6 April 2006
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Please send all your answers to each tutor-marked assignment (TMA), together with a completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

Be sure to fill in the correct assignment number on the PT3 form, and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

Each TMA is marked out of 100. The marks allocated to each part of a TMA question are indicated in brackets in the margin. Your overall score for a TMA will be the sum of your marks for these questions.

You are advised to keep a copy of your assignments in case of loss in the mail. Also keep all your marked assignments as you may need to make reference to them in later assignments or when you revise for the examination.

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10.1

Points to note when preparing solutions to TMA questions

- Contact your tutor if the meaning of any part of a question does not seem clear.
- Your solutions should not involve the use of Mathcad, except in those parts of questions where this is explicitly required or suggested.
- Clarity and accuracy of presentation are important in these assignments, so make sure that you express your answers as precisely as you can, giving detailed explanations where appropriate.
- Where a question involves mathematical calculation, show all your working. You may not receive full marks for a correct final answer that is not supported by working. You may receive some marks for working even if your final answer is incorrect or your solution is incomplete.
- Whenever you perform a calculation using a numerical answer found earlier, you should use the full calculator-accuracy version of the earlier answer to avoid rounding errors.
- Number all of your pages, including any computer printouts.
- Indicate in each solution the page numbers of any computer printouts associated with that solution.
- The marks allocated to the parts of the questions are indicated in brackets in the margin. Each TMA is marked out of 100. Your overall score for a TMA will be the sum of your marks for each question part.

This assignment covers Block A.

Question 1 – 10 marks

You should be able to answer this question after studying Chapter A1.

Find a closed form for the following sequence:

$$u_0 = 7, \quad u_1 = 6, \quad u_{n+2} = -\tfrac{3}{2}u_{n+1} + u_n \quad (n = 0, 1, 2, \ldots). \qquad [10]$$

Question 2 – 10 marks

You should be able to answer this question after studying Chapter A1.

Let u_n be the sequence

$$u_n = 5^n + (-3)^n \quad (n = 0, 1, 2 \ldots).$$

Show that u_n satisfies the identity

$$u_{n+1}u_{n-1} - u_n^2 = 64 \times (-15)^{n-1}, \quad \text{for } n = 1, 2, 3, \ldots. \qquad [10]$$

Question 3 – 15 marks

You should be able to answer this question after studying Chapter A1.

(a) Let F_n be the Fibonacci sequence.

(i) Use the Fibonacci recurrence relation to express both F_{n+3} and F_n in terms of F_{n+1} and F_{n+2}, for $n = 0, 1, 2, \ldots$. [3]

(ii) Use your answer to part (a)(i) to show that

$$F_{n+2}^2 - F_{n+1}^2 = F_n F_{n+3}, \quad \text{for } n = 0, 1, 2, \ldots. \qquad [3]$$

(b) This part is concerned with the identity

$$F_{n+1}^2 - F_{n-1}^2 = F_{2n}, \quad \text{for } n = 1, 2, 3, \ldots, \qquad (*)$$

which you are asked to explore using the methods covered in the Mathcad file 221A1-02.

(i) Use Mathcad to verify that the equation $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ is true for $n = 1, 2, 3, \ldots, 9$. Your answer should consist of a Mathcad printout giving appropriate tables of values for each side of the equation for appropriate values of n. [4]

(ii) Let u_n be the sequence specified by the linear second-order recurrence system

$$u_0 = 0, \; u_1 = b, \; u_{n+2} = u_{n+1} + u_n \quad (n = 0, 1, 2, \ldots),$$

where b is any number. By using Mathcad to experiment with the values of $u_{n+1}^2 - u_{n-1}^2$ and u_{2n} for various values of b, state as a conjecture an identity involving these quantities which is more general than the identity $(*)$. Include Mathcad printouts with tables for three values of b with which you have experimented. [5]

Question 4 – 30 marks

You should be able to answer this question after studying Chapter A2.

(a) This part concerns the conic with equation

$$8x^2 + 9y^2 = 18.$$

(i) By rearranging the equation of the conic to match that of a conic in standard position, classify the conic as an ellipse, parabola or hyperbola, and sketch the curve. [6]

(ii) Find the eccentricity, foci and directrices of this conic, and mark the foci and directrices on your sketch. [5]

(iii) Check your answers to part (a)(ii) by verifying that the equation $PF = ePd$ holds at each of the *two* points P where the conic intersects the x-axis, where F is the focus with negative x-coordinate, d is the corresponding directrix and e is the eccentricity. [4]

(b) Now consider the curve with equation

$$\tfrac{8}{9}x^2 + \tfrac{8}{3}x + y^2 - 6y + 9 = 0.$$

(i) Show that this curve is a conic which can be obtained from the conic in part (a) by translation, and describe the translations required. [5]

(ii) Use your answers to part (a) to sketch this conic, showing its centre, vertices and axes of symmetry, and the slopes of any asymptotes. [4]

(c) (i) Write down parametric equations for the conics in parts (a) and (b). [2]

(ii) Use the parametric equations in part (c)(i) to plot both conics using Mathcad, and provide printouts of your graphs. (You may find it useful to start from the Mathcad file 221A2-01. You can give your answer by plotting both conics on the same diagram if you wish.) [4]

Question 5 – 20 marks

You should be able to answer this question after studying Chapter A3.

(a) Using the exact values of the sine and the cosine of both $\frac{3}{4}\pi$ and $\frac{1}{3}\pi$, and one of the sum and difference formulas, show that the exact value of $\sin(\frac{5}{12}\pi)$ is $\frac{1}{4}(\sqrt{2} + \sqrt{6})$. [5]

(b) (i) Using a half-angle formula, show that

$$\cos^2(2\theta) = \tfrac{1}{2}\cos(4\theta) + \tfrac{1}{2}.$$ [2]

(ii) Using a half-angle formula and the result of part (b)(i), show that

$$\sin^4\theta = \tfrac{3}{8} - \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{8}\cos(4\theta)$$

for any angle θ. [3]

(c) Given that $\sec\theta = 2$, $\tan\theta = -\sqrt{3}$ and $-\pi < \theta < \pi$, find the exact value of the angle θ in radians. Justify your answer. [3]

(d) Given that $0 < \theta < \frac{1}{2}\pi$ and $\sin\theta = \frac{1}{4}$, use appropriate trigonometric formulas to find the exact values of the following.

(i) $\cos(2\theta)$ [2]

(ii) $\cos\theta$ [2]

(iii) $\sin(3\theta)$ (Hint: $3\theta = 2\theta + \theta$.) [3]

Question 6 – 15 marks

You should be able to answer this question after studying Chapter A3.

(a) A triangle has vertices at the points $A(1,3)$, $B(-2,1)$ and $C(2,-1)$. Suppose that the triangle is to be moved so that B is at the origin and BA lies along the positive y-axis. One isometry that achieves this transformation is the composite of a translation followed by a rotation. (You may find it helpful to sketch the triangle.)

(i) Determine the translation that moves B to the origin, giving your answer in the form $t_{a,b}$. Write down a formal definition of this translation in two-line notation. [2]

(ii) Find the images A' of A and C' of C under the translation in part (a)(i). [2]

(iii) Let r_θ be the rotation that completes the required transformation. Find the exact values of $\tan\theta$, $\cos\theta$ and $\sin\theta$, and hence write down a formal definition of r_θ using two-line notation. (There is no need to work out the value of the angle θ.) [5]

(iv) Find the coordinates of the images of A' and C' under the rotation r_θ. Give your answers as exact values. [2]

(v) Write down a formal definition of the composite transformation; that is, the translation in part (a)(i) followed by the rotation in part (a)(iii). (There is no need to simplify your answer.) [2]

(b) In this part of the question you should use a copy of Mathcad file 221A3-02 to obtain the graph of a transformed ellipse.

You should provide a printout showing the graph and the appropriate settings in the worksheet.

Modify the worksheet so that the graph shows the result of applying the rotation $r_{\pi/6}$ to the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$ [2]

This assignment covers Block B.

Question 1 – 15 marks

You should be able to answer this question after studying Chapter B1.

In this question, f is the function

$$f(x) = -\tfrac{1}{5}x^2 + \tfrac{1}{3}x + \tfrac{5}{3}.$$

(a) Use algebra to find the fixed points of f, and to classify them as attracting, repelling or indifferent. [6]

(b) Use the gradient criterion to determine an interval of attraction for one of the fixed points of f. [5]

(c) Find the exact values of the second and third terms of the sequence obtained by iterating f with initial term $x_0 = -3$. Hence state the long-term behaviour of this sequence, explaining your reasoning. [4]

Question 2 – 15 marks

You should be able to answer this question after studying Chapter B1.

In this question, you should use a copy of Mathcad file 221B1-03 to study iteration of the function

$$f(x) = \frac{-x^3 + 2x^2 + 6x - 2}{(1 + x^2)^2}.$$

For each part you should provide printouts illustrating your answers. You should reset the axis limits to $s1 := -3.5$ *and* $s2 := 1.5$ *to obtain your printouts.*

(a) Find the fixed points of f, and classify them as attracting, repelling or indifferent. [6]

(b) Find the 2-cycle of f, and classify it as attracting, repelling or indifferent. [3]

(c) Find the long-term behaviour of the sequence obtained by iterating f with initial term $x_0 = 0$, explaining how you reached your conclusion. [3]

(d) Repeat part (c) with the initial term $x_0 = 1$. [3]

Question 3 – 5 marks

You should be able to answer this question after studying Chapter B1.

(a) By writing down and solving a suitable equation in k, find a positive integer k such that

$$(x^2)^{20-k}\left(\frac{1}{x}\right)^k = x,$$

where x is a variable. [2]

(b) Use the Binomial Theorem and your answer to part (a) to determine the coefficient of x in the expansion of

$$\left(2x^2 - \frac{1}{x}\right)^{20}.$$ [3]

Question 4 – 15 marks

You should be able to answer this question after studying Chapter B2.

Let f be the linear transformation represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}.$$

(a) State what effect f has on areas, and whether f changes orientation. [2]

(b) Find the matrix that represents the inverse of f. [2]

(c) (i) Use the matrix that you found in part (b) to find the image $f(\mathscr{C})$ of the unit circle $\mathscr{C}$ under f, in the form

$$ax^2 + bxy + cy^2 = d,$$

where a, b, c and d are integers. [5]

(ii) What is the area enclosed by $f(\mathscr{C})$? [1]

For part (d), you should provide a printout of your work.

(d) Using page 5 of Mathcad file 221B2-01, express the matrix $\mathbf{M}$ as a product of matrices of basic linear transformations. [5]

Question 5 – 20 marks

You should be able to answer this question after studying Chapter B2.

In this question, f and g are both affine transformations. The transformation f is reflection in the line $y = -x + 1$, and g maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(1,-5)$, $(1,-4)$ and $(0,-5)$, respectively.

(a) Determine g in the form $g(\mathbf{x}) = \mathbf{Ax} + \mathbf{a}$, where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a vector with two components. [3]

(b) Use the approach of Subsection 4.2 of Chapter B2 to find f. That is, write down the matrix that represents reflection in an appropriate line through the origin, and find f (in the same form as for g in part (a)) by first translating an appropriate point to the origin. [6]

(c) Find the affine transformation $g \circ f$ (in the same form as for g and f in parts (a) and (b)). [5]

(d) Hence, or otherwise, find the images of the points $(0,0)$, $(0,-2)$, $(2,-2)$ and $(2,0)$ under $g \circ f$. Mark these points and their images on the same diagram, making it clear which point maps to which. Describe $g \circ f$ geometrically as a single transformation. [6]

Question 6 – 30 marks

You should be able to answer this question after studying Chapter B3.

(a) Without reference to Mathcad, find the eigenvalues and eigenlines of the matrix

$$\mathbf{A} = \begin{pmatrix} 6 & 8 \\ -5 & -8 \end{pmatrix}.$$

For each eigenvalue, give an eigenvector with integer components. [8]

(b) Express the matrix $\mathbf{A}$ in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. You should evaluate $\mathbf{P}$ and $\mathbf{P}^{-1}$. [4]

(c) Use part (b) to find the matrix $\mathbf{A}^5$. [4]

(d) Calculate the second and third points of the iteration sequence with recurrence relation $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ $(n = 0, 1, 2, \ldots)$, for

(i) $\mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$,

(ii) $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. [4]

For part (e), you should provide printouts of your work.

(e) Use Mathcad file 221B3-01 to plot the first four points of each of the iteration sequences in part (d). [4]

(f) Describe the long-term behaviour of each of the iteration sequences in part (d). [6]

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